Reduced-Order Proper $H_\infty$ Controllers for Descriptor Systems:
Existence Conditions and LMI-Based Design Algorithms

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Abstract—In this paper, we present a new approach to investigate the existence and design of reduced-order proper $H_\infty$ controllers that provide the same level of performance as that of full-order controllers. By revealing some special features of the LMI-based solvability conditions for the $H_\infty$ control problem for descriptor systems, we obtain a refined bound on the order of $H_\infty$ controllers, which is independent of (invariant under the allowed transformations on) a descriptor realization of the generalized plant. Moreover, we provide two LMI-based algorithms to design the reduced-order controllers and demonstrate the validity of the presented theoretical results via two numerical examples. This paper not only extends in a satisfying way the results on reduced-order $H_\infty$ controllers for state-space systems to descriptor systems, but also provides insight into the mechanism by which the order of $H_\infty$ controllers for descriptor systems can be reduced through a consideration of the unstable finite zeros or infinite zeros.

I. INTRODUCTION

In many practical applications, the descriptor (also known as implicit, singular, semistate) system description is more convenient and natural than the state-space description. It provides a natural mathematical representation of many practical systems because it is able to describe nondynamic constraints, and finite dynamic and impulsive behavior simultaneously (see [8] and references therein). Regarding the standard $H_\infty$ control, Glover et al. [5] provided a descriptor representation of all the solutions to the four-block general distance problem. To remove the assumptions required in [2] on the infinite and finite $j\omega$-axis zeros of the generalized plant in the state-space model, Hara et al. [6] and Copeland and Safonov [1] employed a descriptor system representation and demonstrated that it is useful for solving the singular $H_\infty$ control problem.

Subsequent studies have dealt with the descriptor $H_\infty$ control problem, which treats the generalized plant in descriptor form [10], [15], [16], [20]. One application is to tackle the mixed sensitivity problem for a physical plant with nonproper weights, because it is often desirable to choose some nonproper weights [9] and because relaxing the requirements of the state-space model of the generalized plant provides more freedom in choosing the weights. Without making any extra assumptions about the direct feedforward matrices in a descriptor realization of the generalized plant, Masubuchi [11], [12] obtained better solvability conditions for the $H_\infty$ control problem for descriptor systems in terms of LMIs and a rank constraint. He also showed that there exists a proper $H_\infty$ controller with an order not greater than rank $E$ for a solvable descriptor $H_\infty$ control problem, see (3) for the definition of $E$.

For practical applications, it is very important to design reduced-order controllers; and a great deal of research has been done on reduced-order $H_\infty$ controllers for state-space systems. Studies employing an LMI-based approach have appeared on the existence and design of reduced-order controllers for the $H_\infty$ control problem with infinite zeros [21], with real unstable transmission zeros [19], and with unstable invariant zeros [22].

This paper is concerned with the existence and design of reduced-order proper $H_\infty$ controllers with an order strictly less than rank $E$ for descriptor systems, a topic that has not received much attention. If we directly apply the approach to analyzing and designing reduced-order $H_\infty$ controllers for state-space systems in [22] to the LMI-based results in [11], [12], we can obtain a bound on the order of $H_\infty$ controllers. However, such a bound is dependent on a descriptor realization of the generalized plant and can not reveal the existence of a reduced-order controller for a particular descriptor realization.

In this paper, we present a new approach to investigate the existence and design of reduced-order proper $H_\infty$ controllers by revealing some special features of the LMI-based solvability conditions for the $H_\infty$ control problem for descriptor systems, see Remark 1 for reference. We provide insight into the mechanism by which the order of a controller can be reduced through a consideration of the unstable finite zeros or infinite zeros. Specifically, we obtain a refined bound on the order of $H_\infty$ controllers, which is expressed in terms of the original parameter matrices of a descriptor realization of the generalized plant. We show that a prominent feature of this bound is the invariance under the allowed transformations [18], which have been widely used in analyzing descriptor systems, on a descriptor realization of the generalized plant. In this sense, this bound is independent of a particular
descriptor realization of the generalized plant. Moreover, we provide two LMI-based algorithms to design the reduced-order controllers; and we demonstrate the validity of the presented theoretical results via two numerical examples.

This paper is organized as follows: Section II gives some preliminary information on descriptor systems, matrix pencils, and the $H_\infty$ control problem for descriptor systems. Section III presents a bound on the order of an $H_\infty$ controller for descriptor systems via a direct approach. Section IV presents a refined bound on the order of an $H_\infty$ controller and a constructive algorithm for designing the reduced-order controllers. Section V shows a linear objective minimization based algorithm for designing the reduced-order controllers. Section VI gives the conclusions. Due to space limitations, we suggest the reader to refer to [23] for some preliminary information on the allowed transformation in [18] and reduced-order $H_\infty$ controllers for state-space systems in [22], for all proofs, and computational detail of the two numerical examples.

Notation

1) $\mathbb{C}$: the open complex plane.
2) $\mathbb{R}^{m \times r}$: the set of all $m \times r$ constant real matrices.
3) $I_n$: identity matrix of size $n \times n$.
4) $A^T$: the transpose of matrix $A$.
5) $H A: A + A^T$ for the square matrix $A$.
6) $B^\perp$: full-row-rank matrix with the maximal number of rows satisfying $B^\perp B = 0$; that is, the rows of $B^\perp$ represent the basis of the left null space of $B$.
7) $X > 0$ ($X \geq 0$, $X < 0$): $X$ is symmetric positive (semi-positive, negative) definite.

II. Preliminaries

A. Descriptor Systems, and Finite and Infinite Zeros of Matrix Pencils

Consider system $\Sigma$ with the following descriptor realization:

$$\Sigma : \begin{cases} E \dot{x} &= Ax + Bu, \\
                    z &= Cx + Dw, \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the descriptor variable, $w$ is the input, and $z$ is the output of the system. Assume that $E \in \mathbb{R}^{n \times n}$ and $-sE + A$ is regular; that is, $\det(-sE + A) \neq 0$. The finite eigenvalues of the pencil $-sE + A$ (which are the roots of $\det(-sE + A) = 0$) are called the finite dynamic modes of $\Sigma$. The infinite eigenvalues of $-sE + A$ are defined to be the zero eigenvalues of $-sA + E$. The infinite eigenvalues corresponding to grade-one infinite generalized eigenvectors, $v_1^i$, that satisfy $Ev_1^i = 0$ are called the nondynamic modes of $\Sigma$. The infinite eigenvalues corresponding to grade-$k$ ($k \geq 2$) infinite generalized eigenvectors, $v_k^i$, that satisfy $Ev_k^i = Av_k^i - 1$ are called the impulsive modes of $\Sigma$. The system $\Sigma$ is admissible if $\Sigma$ has neither any impulsive modes nor any unstable finite dynamic modes.

For the pencil $H(\lambda) = -\lambda K + L$, $\lambda_0 \in \mathbb{C}$ is a finite zero of $H(\lambda)$ if rank $H(\lambda_0) < \text{normal rank } H(\lambda)$, where normal rank $H(\lambda)$ is the rank of $H(\lambda)$ almost everywhere in $\mathbb{C}$. The zero structure of $H(\lambda)$ at infinity is defined as the zero structure of $H(\lambda^{-1})$ at $\lambda = 0$; Verghese et al. [17] concluded that a $k$th-order infinite elementary divisor of a pencil (in the terminology of the Kronecker pencil theory) corresponds to a $(k - 1)$th-order zero at infinity.

B. LMI-Based $H_\infty$ Control for Descriptor Systems

We recall the LMI-based solvability conditions for the $H_\infty$ control problem for descriptor systems in [11], [12]. Consider a generalized plant, $G(s)$, described by

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}, \quad (2)$$

Its descriptor realization is

$$\begin{cases} E \dot{x} &= Ax + B_1w + B_2u, \\
                    z &= C_1x + D_{11}w + D_{12}u, \\
                    y &= C_2x + D_{21}w + D_{22}u, \quad \text{with } D_{22} = 0, \quad (3) \end{cases}$$

where $x \in \mathbb{R}^n$ is the descriptor variable, $w \in \mathbb{R}^{m_1}$ is the exogenous input, $u \in \mathbb{R}^{m_2}$ is the control input, $z \in \mathbb{R}^{p_1}$ is the controlled error, and $y \in \mathbb{R}^{p_2}$ is the observation output. Assume that $E \in \mathbb{R}^{n \times n}$ and $-sE + A$ is regular. Let $r = \text{rank } E$.

Consider a controller, $C(s)$, given by

$$\begin{bmatrix} E \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix}, \quad (4)$$

where $x_c \in \mathbb{R}^{n_c}$ and $E_c \in \mathbb{R}^{n_c \times n_c}$.

For a given $\gamma > 0$, the (suboptimal) $H_\infty$ control problem is to find a control law, $u(s) = C(s)y(s)$, such that the closed-loop system is admissible (stable and impulsive-free) and $\|T_{zw}(s)\|_\infty < \gamma$, where $T_{zw}(s)$ is the closed-loop transfer function matrix from $w$ to $z$.

Lemma 1: [11], [12] For a given $\gamma > 0$, the $H_\infty$ control problem for the generalized plant (3) is solvable if and only if there exist matrices $X \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times n_1}$, and $Z \in \mathbb{R}^{n_1 \times n}$ such that

$$\begin{bmatrix} E^T X & E^T Y \\ E & Y \end{bmatrix} \geq 0, \quad (5)$$

$$E^TW = 0, \quad EZ^T = 0, \quad (6)$$

$$L_B(Y, Z) < 0, \quad (7)$$

$$L_C(X, W) < 0, \quad (8)$$

where

$$L_B(Y, Z) = \begin{bmatrix} B_2 & D_{12} \\ D_{11} & C_1 \end{bmatrix} \begin{bmatrix} AY^T + YA^T & YC_1^T \\ C_1Y^T & D_{11} + ZC_1^T \end{bmatrix} \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \begin{bmatrix} B_2 & D_{12} \\ D_{11} & C_1 \end{bmatrix}.$$
\[ L_C(X, W) = \begin{bmatrix} C_T^T & A^T X + X^T A \\ D_{21}^T & B_T^T X + W^T A \end{bmatrix} C_1 \]

\[ X^T B_1 + A^T W \\
B_1^T W + W^T B_1 - \gamma I \\
D_{11} \]

\[ D_{T1} \]

(10)

If (5)-(8) are satisfied, then an \( H_\infty \) controller exists in the state-space form with order, \( r_c \), satisfying

\[ r_c \leq r_0(X, Y) := \text{rank} \begin{bmatrix} E^T X & E^T Y \end{bmatrix} - \text{rank} E. \]

Using a solution satisfying (5)-(8), Masubuchi [11, 12] provided a synthesis method for obtaining \( E_c, A_c, B_c, C_c \), and \( D_c \) in (4) such that \( \text{rank} E_c = r_0(X, Y) \) and \( -sE_c + A_c \) is impulsive-free by solving an LMI and, if necessary, adding a small perturbation to the solution.

Owing to \( r_0(X, Y) \leq \text{rank} E = r \), there exists an \( H_\infty \) controller in the state-space form with order, \( r_c \), satisfying \( r_c \leq r \); however, finding a solution satisfying (5)-(8) and \( r_0(X, Y) < r \), in general, is non-convex. Therefore, we will study the existence conditions and LMI-based design algorithms for reduced-order proper \( H_\infty \) controllers with order strictly less than \( r \), mainly by exploiting LMI (7), and we can do dually by exploiting LMI (8).

**III. A Bound on the ORDER of \( H_\infty \) Controllers: A DIRECT APPROACH**

Without separating the matrix variables \( E^T Y \) and \( E^T X \), which determine the order of the \( H_\infty \) controller (see (11)), from other matrix variables in the LMIs of Lemma 1, we obtain the following proposition via a direct application of a previous result on the analysis and design of reduced-order \( H_\infty \) controllers for state-space systems in [22].

**Proposition 1:** For a given \( \gamma > 0 \), suppose that the \( H_\infty \) control problem for the generalized plant (3) is solvable. Then, there exists a proper \( H_\infty \) controller whose order, \( r_c \), satisfies

\[ r_c \leq n_{\text{so}} := \min \left\{ \min_{\text{Re} [\lambda] \geq 0} \rho_0 (\lambda), \rho_{\infty} \right\} \leq r, \]

where \( \text{Re} [\lambda] \) denotes the real part of \( \lambda \in \mathbb{C} \), and

\[ \rho_0 (\lambda) := \text{rank} \begin{bmatrix} -\lambda E + AE^+ E & B_2 \\ C_1 E^+ E & D_{12} \end{bmatrix} - \text{rank} \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}, \]

\[ \rho_{\infty} := \text{rank} \begin{bmatrix} E & B_2 \\ 0 & D_{12} \end{bmatrix} - \text{rank} \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}, \]

where \( E^+ \) is the Moore-Penrose pseudoinverse of \( E \).

For the special case \( E = I_n \), Proposition 1 reduces to the one in [22] for state-space systems. However, as we explain in what follows, \( n_{\text{so}} \) in Proposition 1, is dependent on a particular descriptor realization of the given transfer matrix \( G(s) \) in (2). First, for nonsingular matrices \( T \) and \( P \), since in general \( (TEP)^+ \neq P^{-1}E^T P^{-1} \), \( n_{\text{so}} \) in (12) is, in general, variant under the restricted system equivalence transformations [13], [18] (p. 814) on (3).

Second, we illustrate that the values of \( n_{\text{so}} \) are different for two descriptor realizations of \( G(s) \) which can be transformed to each other via some sequence of the allowed transformations [18] (p. 817). Consider a proper \( G(s) \), for which a state-space realization is given in (3) with \( E = I_n \). By introducing an additional descriptor variable in two different ways, we obtain the following two descriptor realizations for \( G(s) \):

\[ \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 & B_2 \\ 0 & 0 & -I_{m_1} & I_{m_1} \\ C_1 & 0 & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ y \end{bmatrix} \]

(15)

\[ \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 & B_2 \\ 0 & 0 & -I_{m_2} & I_{m_2} \\ C_1 & 0 & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ y \end{bmatrix} \]

(16)

Using (12), we know that \( n_{\text{so}} \) for realization (15) is the same as the bound in [22]. However, we can see that \( n_{\text{so}} \) for realization (16) is

\[ n_{\text{so}} = \min_{\text{Re} [\lambda] \geq 0} \text{rank} \begin{bmatrix} -\lambda I_n + A \\ C_1 \end{bmatrix}, \]

(17)

which is greater than or equal to the bound in [22].

Thus, for a given descriptor realization, the bound in Proposition 1 may fail to reveal the existence of a reduced-order controller. Indeed, from the two numerical examples in this paper, we can see that a reduced-order controller cannot be shown to exist by using Proposition 1. Let us consider the following generalized plant, \( G(s) \), taken from [15], which considered the \( H_\infty \) control problem for \( G^T(s) \) with \( \gamma = 1 \).

\[ G(s) = \begin{bmatrix} 2s & s^2 + 2s - 1 \\ 2s + 1 & s + 1 \end{bmatrix}. \]

(18)

A descriptor form representation of this plant is

\[ E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]

\[ A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \]

\[ B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]

\[ C_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \]

\[ D_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, D_{12} = 1, D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}. \]

Takaba et al. [15] gave the following controller, for which \( ||T_{zw}||_\infty = 0.934 \):

\[ C(s) = \frac{-s + 1.807}{2(s + 1.301)(s + 2.043)}. \]

(19)
Since $G_{12}(s)$ has an unstable transmission zero at $s = -1 + \sqrt{2}$, based on the fact of controller order reduction owing to an unstable invariant zero of $G_{12}(s)$ for the $H_\infty$ control problem for state-space systems, we expect the existence of a reduced-order controller for (18). However, from (12) in Proposition 1, we obtain $n_{160} = 2$, which fails to show that a reduced-order controller exists.

Therefore, a direct application of the result for the state-space realization to the descriptor system is not good enough; and further investigation is needed to obtain a refined result, which is discussed in the next section.

IV. A REFINED BOUND ON THE ORDER OF $H_\infty$ CONTROLLERS AND A DESIGN ALGORITHM

A. A Refined Bound on the Order of $H_\infty$ Controllers

By revealing some special features of the LMI-based solvability conditions for the $H_\infty$ control problem for descriptor systems, we can separate $EY^T$ and $E^TX$, which determine the order of the $H_\infty$ controller, from other matrix variables in the LMIs of Lemma 1. This yields more freedom in the choice of matrix variables $EY^T$ and $E^TX$ for obtaining a refined bound on the order of $H_\infty$ controllers. This bound is expressed in terms of the original parameter matrices of a descriptor realization of the generalized plant, and is invariant under the allowed transformations on the system matrix of the descriptor realization. First, we present the following theorem.

Theorem 1: For a given $\gamma > 0$, suppose that the $H_\infty$ control problem for the generalized plant (3) is solvable. Then, there exists a proper $H_\infty$ controller whose order, $r_c$, satisfies

$$r_c \leq \tilde{n}_b := \min \left\{ \min_{\text{Re}[\lambda] \geq 0} \rho(\lambda), \rho_\infty \right\},$$

where

$$\rho(\lambda) := \text{rank } E + \text{rank} \begin{bmatrix} -\lambda E + A & B_2 \\ C_1 & D_{12} \end{bmatrix} - \text{rank} \begin{bmatrix} E & 0 \\ A & B_2 \\ C_1 & D_{12} \end{bmatrix},$$

$$\rho_\infty := \text{rank} \begin{bmatrix} E & 0 \\ A & B_2 \\ 0 & C_1 & D_{12} \end{bmatrix} - \text{rank} \begin{bmatrix} E & 0 \\ A & B_2 \\ 0 & C_1 & D_{12} \end{bmatrix}.$$ (22)

Moreover, the following inequalities hold:

$$\rho(\lambda) \leq \rho_0(\lambda) \leq r \text{ for } \lambda \in \mathbb{C},$$

$$\rho_\infty \leq \tilde{\rho}_\infty \leq r, \text{ } n_b \leq n_{160},$$

where $\rho_0(\lambda)$, $\rho_\infty$, and $n_{160}$ are defined in (13), (14), and (12), respectively. ☐

For the special case $E = I_n$, Theorem 1 reduces to the one in [22] for state-space systems. Moreover, the values of $n_b$ for the two descriptor realizations (15) and (16) are the same as the one in [22] for state-space systems. Next, we present the following theorem to reveal a prominent difference between $n_{160}$ in (12) and $n_b$ in (20).

Theorem 2: For the generalized plant $G(s)$ with its descriptor realization in (3), the inequality in (20) holds with $n_b$ invariant under the allowed transformations on the system matrix of (3).

We present a dual result of Theorems 1 and 2.

Theorem 3: For a given $\gamma > 0$, suppose that the $H_\infty$ control problem for the generalized plant (3) is solvable. Then, there exists a proper $H_\infty$ controller whose order, $r_c$, satisfies

$$r_c \leq \tilde{n}_b := \min \left\{ \min_{\text{Re}[\lambda] \geq 0} \tilde{\rho}(\lambda), \tilde{\rho}_\infty \right\},$$

where

$$\tilde{\rho}(\lambda) := \text{rank } \begin{bmatrix} -\lambda E + A & B_1 \\ C_2 & D_{21} \end{bmatrix} - \text{rank} \begin{bmatrix} E & A & B_1 \\ 0 & C_2 & D_{21} \end{bmatrix},$$

$$\tilde{\rho}_\infty := \text{rank} \begin{bmatrix} 0 & E & 0 \\ E & A & B_1 \\ 0 & C_2 & D_{21} \end{bmatrix} - \text{rank} \begin{bmatrix} E & A & B_1 \\ 0 & C_2 & D_{21} \end{bmatrix}.$$ (27)

Moreover, the inequality in (25) holds with $\tilde{n}_b$ being invariant under the allowed transformations on the system matrix of (3). ☐

The following corollary yields from Theorems 1 and 3.

Corollary 1: Suppose the $H_\infty$ control problem for the generalized plant (3) is solvable. If the system matrix of $G_{12}(s)$ or $G_{21}(s)$ induced from (3), which is

$$\Gamma_{12}(s) := \begin{bmatrix} -sE + A & B_2 \\ C_1 & D_{12} \end{bmatrix},$$

$$\Gamma_{21}(s) := \begin{bmatrix} -sE + A & B_1 \\ C_2 & D_{21} \end{bmatrix},$$

has unstable finite zeros or infinite zeros, then there exists a proper $H_\infty$ controller whose order, $r_c$, is strictly less than rank $E$. ☐

The system matrices of $G_{12}(s)$ and $G_{21}(s)$ play an important role in GARE (generalized algebraic Riccati equation) based solutions of the $H_\infty$ control problem for the descriptor system [15], [20]. However, their role is somewhat unclear in the LMI-based approach to this problem. We clarified their role in LMI-based solutions by establishing the relationship between the bound on the order of the $H_\infty$ controller, and the (unstable finite or infinite) zeros of the system matrices.

Finally, we present a remark to reveal some special features of the LMI-based solvability conditions for the $H_\infty$ control problem for descriptor systems in [11], [12].
Remark 1: Using the projection lemma for variable elimination, we can show that the LMI-based solvability conditions in Lemma 1 are equivalent to the negative definiteness of ten constant matrices, and the feasibility of three LMIs which contain only the matrix variables determining the controller order and have the same forms as those for the $H_\infty$ control problem for state-space systems.

B. A Constructive Design Algorithm

Based on Theorem 1, we propose Algorithm 1 which is a constructive design algorithm for designing a reduced-order $H_\infty$ controller. Due to space limitations, we suggest the reader to refer [23] for Algorithm 1.

We apply Theorem 1 and Algorithm 1 to the generalized plant (18) taken from [15], for which the unstable finite zero at $s = \lambda_0 = -1 + \sqrt{2}$. From (20) in Theorem 1, we have $n_k = \text{rank} \rho(\lambda_0) = 1$. This means that a first-order $H_\infty$ controller exists. Using Algorithm 1 with the aid of the LMI control toolbox [4], we obtain the following first-order controller:

$$C(s) = \frac{-0.4152(s + 1.643)}{s + 1.721}.$$  

(30)

Under this controller, the closed-loop system is impulse-free and has the poles $-3.2946, -1.7569$, and $-1.0000$; and $|T_{zw}|_{\infty} = 0.8029$. In addition, we obtain $\gamma_{\text{opt}} = 0.7678$, where $\gamma_{\text{opt}}$ is the optimal $H_\infty$ performance for $G(s)$.

V. AN EFFICIENT ALGORITHM FOR DESIGNING REDUCED-ORDER $H_\infty$ CONTROLLERS

For the $H_\infty$ control problem for state-space systems, Skelton et al. [14] (p. 167) suggested the use of min trace $(X + Y)$ in combination with the three LMI as a heuristic method of constructing a reduced-order $H_\infty$ controller; and Xin [22] proved that such linear objective minimization always yields a controller of order not greater than the bound determined by the unstable invariant zeros or infinite zeros of $G_{12}(s)$ or $G_{21}(s)$. We can extend this result to descriptor systems as shown in the following theorem.

Theorem 4: For a given $\gamma > 0$, suppose that the $H_\infty$ control problem for the generalized plant (3) is solvable. Then, there exist scalars $\epsilon_B > 0$ and $\epsilon_C > 0$ such that

$$L_B(Y, Z) + \epsilon_B I \leq 0, \quad L_C(X, W) + \epsilon_C I \leq 0,$$  

(31)

where $L_B(Y, Z)$ and $L_C(X, W)$ are defined in (9) and (10), respectively. Let

$$(X_m, Y_m, W_m, Z_m) = \arg \min_{X, Y, W, Z} \text{trace}(E^T X + EY^T),$$

subject to (5), (6), and (31).

Then, $X_m, Y_m$, and $n_k$ in (20) satisfy

$$\text{rank} \begin{bmatrix} E^T X_m & E^T Y_m \\ E & EY_m^T \end{bmatrix} - \text{rank} E \leq n_k.$$  

(33)

Note that, if (7) and (8) rather than (31) are used in the optimization problem (32), then the optimal solution may lie on the boundary of $L_B(Y, Z) < 0$ or $L_C(X, W) < 0$.

We provide an algorithm for designing a reduced-order $H_\infty$ controller based on Theorem 4.

Algorithm 2

1. Solve (5)–(8) to obtain a solution, which is denoted $(X_p, Y_p, W_p, Z_p)$.
2. Choose the scalars $\epsilon_B$ and $\epsilon_C$ such that $0 < \epsilon_B < \epsilon_{\text{min}}(-L_B(Y_p, Z_p))$ and $0 < \epsilon_C < \epsilon_{\text{min}}(-L_C(X_p, W_p))$.
3. Solve (32) to obtain $(X_m, Y_m, W_m, Z_m)$.
4. Construct a reduced-order proper controller using the algorithm in [11], [12], and the solution $(X_m, Y_m, W_m, Z_m)$.

We apply Algorithm 2 to the following $G(s)$, for which the $H_\infty$ control problem was studied in [11], [12]:

$$G(s) = \begin{bmatrix} s^2 + 3s + 5 & 0 \\ s^2 + 2s + 3 & s^3 + 8s^2 + 14s + 4 & 0 \\ \frac{s^2 + 2s + 3}{s^2 + 2s + 3} & \frac{s^2 + 8s^2 + 14s + 4}{(s^2 + 2s + 3)(s^2 + 5s + 2)} \end{bmatrix}.$$  

with E-matrix of a descriptor form representation of this plant being $E = \text{diag}(I_5, 0)$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_{1}^T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2^T = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$  

$$D_{11} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Masubuchi [11], [12] stated that $\gamma_{\text{opt}} = 1.7064$ provides the optimal $H_\infty$ performance.

For the descriptor form representation of this plant, from (12), we obtain $n_{\text{so}} = \text{rank} E = 5$. This fails to show that a reduced-order controller exists. Notice that $G_{12}(s)$ and $G_{21}(s)$ are both proper and have neither finite zeros nor infinite zeros; and note that $G_{11}(s)$ and $G_{22}(s)$ are stable. This means that the system matrices of $G_{12}(s)$ and $G_{21}(s)$ have no unstable finite zeros. However, since $G_{22}(s)$ is improper (that is, it has a pole at infinity), the system matrices of $G_{12}(s)$ and $G_{21}(s)$ inherit that pole as their
infinite zero. From (20)–(22) in Theorem 1, we obtain \( n_b = 4 < 5 \). This shows that a reduced-order \( H_\infty \) controller exists.

Setting \( \gamma = 1.8 > \gamma_{opt} = 1.7064 \) and using Algorithm 1, we obtain the following fourth-order controller:

\[
C(s) = \frac{-0.001(s - 315.4)(s + 4.362)(s - 1.74)(s + 0.4383)}{(s + 4.226)(s + 0.3838)(s^2 + 2.844s + 2.192)},
\]

under which the closed-loop system is impulsive-free and has the poles \(-721.28, -4.5576, -4.4998, -1.7756 \pm 1.6896i, -1.2197 \pm 0.2362i, -0.4384, \) and \(-0.3693; and \( ||T_{zw}||_\infty = 1.7294 \).

Now, by using Algorithm 2 with \( \epsilon_B = \epsilon_C = 0.02 \), we obtain \( (X_m, Y_m, W_m, Z_m) \), with the largest eigenvalue of \( X_{m11} - Y_{m11}^{-1} \) being 0.8647 and the others being less than \( 8 \times 10^{-7} \). This yields the following first-order controller:

\[
C(s) = \frac{-0.0344(s + 3.899)}{s + 0.2612}.
\]

Under this controller, the closed-loop system is impulsive-free and has the poles \(-33.654, -4.5701, -1.0369 \pm 1.3359i, -0.4433, \) and \(-0.2337; and \( ||T_{zw}||_\infty = 1.7489 < 1.8 \). For this particular plant and the performance \( \gamma = 1.8 \), minimizing the trace of the linear combination of the matrix variables yields a first-order \( H_\infty \) controller for which the order is less than that determined by the bound \( n_b \).

VI. CONCLUSIONS

In this paper, we have investigated the existence and design of reduced-order proper \( H_\infty \) controllers for descriptor systems with the same level of performance as that of full-order controllers. Using the projection lemma \([3, 7]\) for variable elimination, we \textit{separated} the matrix variables determining the order of the \( H_\infty \) controller \textit{from} other matrix variables in Lemma 1. This yields more freedom in choosing the matrix variables determining the controller order to obtain the refined bound in (20) on the controller order, which is expressed in terms of the original parameter matrices of the system matrix of \( G_{12}(s) \). We showed that a prominent feature of this bound is the invariance under the allowed transformations on a descriptor realization of the generalized plant. When the \( H_\infty \) control problem for descriptor systems is solvable, a reduced-order controller can be shown to exist if either of the system matrices of \( G_{12}(s) \) and \( G_{21}(s) \) has unstable finite zeros or infinite zeros. Two numerical examples demonstrated the validity of the theoretical results obtained in this paper. In this paper, we not only extended in a satisfying way the results on reduced-order \( H_\infty \) controllers for state-space systems to descriptor systems, but also provided insight into the mechanism by which the order of \( H_\infty \) controllers for descriptor systems can be reduced.

REFERENCES